New framework for determining physical properties - I: properties of leptons

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Equations have been found which explain some of the physical properties of the leptons, to a decimal accuracy matching that of their measurement. These equations set logical patterns and can be explained in terms of easy to understand three dimensional geometric pictures. This mathematical framework does not develop from any previous theory but from observation and does contain several new mathematical constants. Anyone familiar with the quantum mechanical description of the hydrogen electron shells can immediately use these equations to verify the masses of the three leptons; the electron e with $m_e = 9.109,389,7(10^{-31})$ kg, the muon μ with $m_\mu = 1.883,532,7(10^{-28})$ kg, and the tau τ with $m_\tau = 3.167,88(10^{-27})$kg. Additionally, the charge of the leptons, $e = 1.602,177,33(10^{-19})$ C, can now be understood as arising from the curvature of certain energy structures for these particles, when formulated in terms of vectors. The mathematical framework also predicts the possibility of a new fourth lepton particle.

Keywords: leptons, elementary charge, particle masses, particle structures, computational physics, applied mathematics

I. INTRODUCTION

A. Objective & Scope

The general objective of this paper is to show observed organizing principles and patterns for the basic subatomic particles. The masses of the quarks are only loosely measured to a few decimals and these particles appear to have added inherent complexities, such as color, that the leptons do not have. The masses of the neutrinos have not been measured at all, but have only had upper limits set for them. The charge and masses of the leptons have been measured to many decimals of accuracy and these particles only have the complexity of two parameters, mass and charge. Thus the specific objective of this work is to demonstrate a mathematical framework that allows the determination of the physical properties of the three leptons; electron e, muon μ , and tau τ.

This work is best described as a mathematical framework and demonstration. This framework does not specifically support any particular theory and is not derived from theory. While there may be implications concerning particle theories indirectly supported by this work, only those conclusions directly supported by the equations found will be reported. The equations found are simply presented, not derived, nor proven in the formal sense of those words.

B. Background

The Standard Model of particle physics has been overwhelmingly verified and yet there are still major outstanding questions. The Standard Model has not and due to its nature can never explain the masses of the elementary particles such as the neutrinos, leptons, and quarks [1,2]. Likewise, the Standard Model cannot explain the occurrence of multiple generations of particles, whether this number is 2, 3, 4, or anything greater than one [1]. Thousands of high energy particle physicists have worked on these problems and searched for ways to explain them by extending the Standard Model. This is by including it as a subset of broader theories such as those of supersymmetry [2], superstring, or supergravity. Unfortunately, despite these efforts involving millions of research hours, this most basic and first measured property of all subatomic particles, mass, still remains unexplained.

There are several major and profound differences between this work and those mathematical physics endeavors which have been reported in the mainstream physics journals.

Almost all of the past and current work to explain physical property data of the subatomic particles, such as their masses, starts from a theoretical platform. A theory or a new slant on an existing theory is proposed. Theoreticians then work downward by developing equations from the theory. These equations are made specific by the insertion of unique specifying constants into them. Then lastly the wealth of particle data is screened for those occurrences which make relatively close matches to the values which have been calculated from the theoretically based equations. Specifically, this "theory first" approach is true for all those approaches to calculating the particle masses which are based in symmetry (supersymmetry), group theory, and matrices. There are a plethora of examples, with [3-9] being typical. In contrast, this work did not assume as its starting point that any particular theory or branch of mathematics, such as set theory or modern algebras, applied to the masses of the particles. The objective of this work was to explain the masses of a class of particles, not to prove, bolster, or support a particular theoretical model of the particle universe.

A majority of researchers assume the existence of some

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hypothetical particles as a necessary starting ingredient to explain the masses of the known ones. This is true for all the work based in supersymmetry [10-14]. Many of these assumed particles will realistically never be provable. The objective of the work here was to explain some of the observed properties of long accepted and undisputable particles, the leptons. The assumption of hypotheticals was prohibited in this work. Assuming unknown entities would have completely self defeated the purpose of this work. If incidentally, after the fact, additional particles were predicted, so be it.

Aside from the theoretical starting point and approach, or lack of it in the case of this work, there are major differences between the scope of this work and that of most current particle research. For example, many researchers mix calculations and discussions of mixing angles in the same work with investigations of particle masses [6,8,13-19]. A more limited scope, only masses and not mixing angles were the targets of the calculations of this work. Additionally after the fact, this work shows that only with a prior knowledge of a structure, either discovered or assumed, will mixing angles probably ever make any sense.

Many researchers mix the study of the lepton masses in with that of the quark masses [4,6,13,15-21]. This broad scope assumes that the masses of the two classes have some definable relation and further usually forceably intertwines their mathematics. Frequently the objective of such work is to use the masses of one or both of these two classes as a means to indulge in the latest hot topic, predicting the masses of the neutrinos [4,6,8,13,15-16,18-20,22-24]. Under the principle of keep it simple, the work here focused exclusively on the masses of the leptons. The masses of the leptons were to be explained, for the leptons themselves, and not for some greater ulterior motive.

As to the nuts and bolts of the calculations themselves, there are yet still more major differences between this work and that of most the research reported in the major journals. Calculations based in symmetry, group theory, and matrices [3-5,7-9,13,16,18] tend to yield many mathematical terms, often dozens, but none specifically tied to any structural properties. We can ask, what physical property holds or embodies each of the many mathematical terms? In contrast this work required that every mathematical factor or term have a plausible explanation assigning it to some observable or self evident mathematical/physical structural feature of the particle. Frequently, calculational physics uses 3 X 3 matrices to predict the lepton masses, or two sets of 3 X 3 matrices for the two families of quarks. Then immediately the six off diagonal elements are rationalized away or discarded [3-5,7-9,13,16,18]. This work is much more straight forward and sticks strictly with the three objects or mathematical forms of discussion. This work does not theorize an excess and then have to find some way to eliminate things. Further this work does not mathematically lock in that there may only be three leptons.

Finally some researchers did use creative, non-theoretical, correlative or numerical based approaches to the particle masses [18-19]. Some even used exponential or logarithmic based calculations as will be found in this work [4,16-17,25]. Unfortunately all these researchers stopped short with only weak correlations for the lepton masses [15,26-27]. The results of such correlations were often only good to one decimal place. The work here was only considered complete and its objective accomplished when the calculations matched the masses of the particles, or other physical property, to that of their measured decimal accuracy!

II. OUTLINE OF WORK

A. Fundamental Approach & Assumptions

The primary unique feature of this work, to explain the masses of the leptons, was to start with the data and work upward toward any generalizing principles which may have become obvious. First mathematical-geometric correlations for the masses of the leptons were to be found. Correlation constants were to be added to make actual predictive equations. Then last these equations were to be placed within any broader body of mathematics of which they may be a part. By following this path it was felt that those theories which may be applicable would become self evident. Additionally this approach guaranteed that any applicable theories would be linked to the data and that the exact equation path of this linkage would be defined.

Regardless of whether a “theory first”or a “data first” correlative approach is used, some assumptions are necessary to set a context or framework for the work. For this work we assumed that the subatomic particles are wave structures, and that these structures are responsible for their many observed and measured properties. Stated as the following hypothesis;

1. All objects in the consensus physical world have a form or structure. This includes the elementary particles which are the objects of discussion of physics.

2. There are no formless particles nor any particles that are mathematical points.

3. The form or structures of the basic objects of the physical universe, subatomic particles, can be described by appropriate mathematical-geometric equations.

4. Further these wave structures or “objects” can be described via mathematical-geometric equations not just in general, but precisely, to as many decimals as necessary.

The three forces applicable to the leptons were assumed to be a-priori to all else. That is; the values of G, μ_0, and ϵ_0 were to be the only basic starting values. The values used are listed in Table VIII in the Appendix. These ultimately would be used to scale or bridge from the world of pure mathematical equations and geometry to the scale of the consensus world of physics and humans. The remaining basic values found in physics reference books such as e, h, α were felt to be derivables, and would only be used in the scaling of the lepton masses, should they become necessary and not create circular references or circular derivations. It turns out e and α are necessary.

Stated differently, the six basic forces; the unary set gravity, the binary set electro-magnetism, and the ternary set red-green-blue do not depend on the particles for their values, but the particles definitely require these forces for their geometric structures. That is; gravity does not require an electron, but an electron depends on gravity for its existence. The implications of this decision will be seen in the discussion of the accuracy of the equation describing the charge of the leptons.

B. Key Mathematics Used

To explain our objective particle properties of charge and mass we found that two bodies of mathematics were needed. Vector mathematics in rectilinear coordinates was needed to describe what could be called the encapsulated electromagnetic force or the entrapped energy of the particle as charge in coulombs. Regular or scalar mathematics in polar coordinates was needed to describe what could be called the stabilized gravitational force or the contained energy of the particle as mass in kilograms. Since these two mathematical descriptions are just two different conceptual views of the same objects, as expected we found many features in common between the two descriptions.

We return to the last time when scientists had thoroughly inundated themselves with the discovery of a zoo of basic particles, the elements of the periodic chart. Then humans were forced once again to conceptualize about things that they could not directly physically examine. A sense of order was restored by the development of quantum mechanics. In that application of mathematics to explain a wealth of physical phenomena, second order differential equations and their solutions as the Laguerre and Legendre orthogonal polynomials became the working tools. These two series explained the patterns which described the repeating rows and columns of the periodic chart. Here we have found that it is again orthogonal polynomials, the Laguerre series, which brings order or mathematical sense to a repeating family of elementary object masses.

Since mathematically the Laguerre polynomials are an open ended series we examined higher members past where our pattern for the electron, muon, and tau stopped. We found that a low energy fourth member of the lepton family was mathematically possible, although it probably has an extremely short half life due to angular instabilities. After this our mathematics indicated that further members of the series resulted in negative masses or unstable energy patterns.

C. Geometric Appearance of the Leptons

The general geometric appearance found for the leptons in this work is that of a toroidal coil; mathematically a cylindrical helix, which is wrapped around into a circle to form the outline of a donut, mathematically a torus. This appearance is the same as that of a photon which instead of propagating linearly through space-time, has its head wrapped around and connected to its tail, and thus goes forever in a tight little circle.

Viewing this geometric picture more rigorously, the mass density has two spatial dimensions, one radial-planar, and one angular. It has two temporal dimensions, one radial-planar, and one angular. The radial spatial and temporal planes are simultaneous or co-planar. The angular spatial and temporal dimensions are at right angles to each other. Thus the net figure in space and time is the three dimensional toroidal coil as described.

The general toroidal coil description of the mass density appearance of the leptons can be made mathematically specific, using 3 dimensional vector notation of an implicit function in time, as follows. Consider a generalized cylindrical figure

$$\mathbf{R}(t) = aCos[F(t)]\mathbf{i} + aSin[F(t)]\mathbf{j} + bF(t)\mathbf{k} \qquad (1)$$

If $F(t) = \lambda t$, then we simply have the cylindrical helix; the outline of an open ended, unlimited, or moving figure. **This is the energy pattern of the photon**. If $F(t) = d_1Cos(e_1t)$, then we have the outline of a cyclic, bounded, or stationary figure, the torus or a toroidal coil. Viewing $d_1Cos(e_1t)$ as a trigonometrically substituted Chebyshev $T_1^\dagger$polynomial, we have the energy pattern of the electron. If $F(t) = T_3^\dagger[d_3Cos(e_3t)]$ and $T_5^\dagger[d_5Cos(e_5t)]$, then we have the general energy patterns of the muon and tau respectively.

D. Mathematical Framework for Mass Density Equations of Leptons

The neutrinos, leptons, and quarks of physics are of a scale so small that they are completely out of touch with not only the human senses, but also with all of the machine extensions of our senses. These “objects” are so out of scale with the human ability to sense and measure that scientists held for many years that these particles are mathematical points, dimensionless, formless, and structureless. Unfortunately this view eliminated all of the common practical mathematical avenues for analyzing these particles. Here by assuming the particles are wave patterns or forms we found relatively simple mathematics could be used to describe them. In fact the mathematical framework found is not only conceptually simpler than the quantum mechanic framework of the periodic chart but at most requires only a knowledge of second semester calculus to follow.

The best known mathematical series describing components of the physical world is the periodic chart of the elements of chemistry. In this setting quantum mechanics offered a conceptual tool to describe the existence of things that were many orders of magnitude out of scale with the human form. This mathematical field gave descriptions not just in general, but gave very exact predictions of the nature of how forms behaved at the scale of concern. Here we have again found that mathematics strikingly similar to quantum mechanics explains an even smaller scale of forms, the leptons. The major elements of the mathematical framework that we found for the leptons are as follows;

1. Radial (2 dimensional, planar) mass density equations, based on the Laguerre orthogonal polynomials.

2. Angular (a single angle) mass density equations, based on the Chebyshev $T^\dagger$ orthogonal polynomials.

3. A series of shells for the higher members of the series, based on Laguerre polynomial derivatives.

4. Embedded or implicit temporal parameters in both the radial and angular equations, as two independent temporal variables.

5. Initial temporal conditions for both the radial and angular equations, which lead to initial multiplying factors or constants.

6. A general scale factor or correlation constant for all the particles, composed of basic a-priori measured physical constants.

7. Several specific scale factors for each member of the series. These factors set definite patterns or form series themselves.

8. An overall equation combining the radial equations, angular equations, and the final scale factors as multipliers.

III. MATHEMATICAL PRELIMINARIES

The generalized cylindrical figure

$$\mathbf{R}(t) = aCos[F(t)]\mathbf{i} + aSin[F(t)]\mathbf{j} + bG(t)\mathbf{k} \tag{2}$$

is important to this work. For such a vector the curvature κ and the torsion τ are rigorously calculated as

$$\kappa = \frac{|\mathbf{R}'(t) \times \mathbf{R}''(t)|}{|\mathbf{R}'(t)|^3} \tag{3}$$

$$\tau = \frac{\left[\mathbf{R}'(t) \times \mathbf{R}''(t) \bullet \mathbf{R}'''(t)\right]}{|\mathbf{R}'(t) \times \mathbf{R}''(t)|^2} \tag{4}$$

and as such both are scalar quantities.

Calculating the quantities $\mathbf{R}'(t)$, $\mathbf{R}''(t)$, $\mathbf{R}'''(t)$, κ , and τ of this most general form of $\mathbf{R}$(t) results in messy and irreducible expressions in both F(t) and G(t) for both κ and τ . If we make the simplifying assumption that F(t) = G(t), then the results are the simple expressions;

$$\kappa = \frac{a}{a^2 + b^2} \tag{5}$$

$$\tau = \frac{b}{a^2 + b^2} \tag{6}$$

Thus with this one restriction, that the implicit function F(t) = G(t), we find that the curvature and the torsion of this generalized cylindrical figure are numerical constants, independent of the implicit variable t and all functions of F(t). In formalized mathematical or quantum mechanic jargon, we would say that this Curvature Operator is invariant under rotation, translation, substitution of F(t), et cetera. This invariance is of prime importance in the equation found which describes the charge of the leptons. The generic mathematical form $a/(a^2 + b^2)$ arises several times in the equations found which calculate the mass densities of the leptons, as well as in that of their charge.

Equally important, we find there appears to be no other mathematical quantity which remains constant as F(t) changes. The unit tangent $\mathbf{T}$, principal normal $\mathbf{N}$, and binormal $\mathbf{B}$ vectors all vary for the circular helix. None of the classical differential operators are satisfactory, regardless of whether they operate on scalars or on vectors. The gradient ∇, both the scalar and vector Laplacians ∇^2, the divergence $\nabla\bullet$, and the curl $\nabla\times$ all remain functions of t or F(t), and most still have unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ embedded in their formulas. Additionally we need to consider the meaning of any candidate quantities. The curvature κ and the torsion τ both refer to the curve or curved surface itself. The unit vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ refer to something which is perpendicular to the surface. The elementary differential operators refer to fields perpendicular to the surface, fluxes through the surface, circulations in the surface, et cetera.

Equally of interest we find there can be 4 combinations of κ and τ if we vary the signs of a and b. If F(t) is a trigonometric Cos() or Sin(), then there can be 2 directions of travel around the toroidal coil appearance or 2 means of "revolving" about the center of the donut, according to whether b is positive or negative. Likewise, there can be 2 directions of rotation or spins about the centerline of this axis of revolution, according to whether a is positive or negative. These 4 combinations, possibly applicable to particles and their anti-particles, need to be reduced to 2 if we allow space to freely rotate about the "objects" described by the vector $\mathbf{R}$(t). By varying the signs of a and b we can now see a means of directly relating observed properties such as charge and hand to mathematical features of proposed wave structures for these particles.

For the purposes of this work the equation for the cylindrical figure needs to be generalized one step further, by specifying the explicit or external functions as follows;

$$\mathbf{R}(t) = aT_n^\dagger(Cos[F(t)])\mathbf{i} + aT_n^\dagger(Sin[F(t)])\mathbf{j} + bF(t)\mathbf{k} \tag{7}$$

For this form we then find the curvature for odd n to be

$$\kappa = \frac{n^2 a}{n^2 a^2 + b^2} = \frac{a}{a^2 + b^2/n^2} \tag{8}$$

IV. APPLICATIONS TO PHYSICAL PROPERTY DETERMINATIONS

A. Charge of the Leptons

The equation which describes the means of calculating the charge of the leptons is quite simple. The origin of the mathematical quantities involved is explainable from vector geometric considerations, although the exact meaning of these quantities is still open to discussion. As a simple starting point, an analogy is given of another physics constant, h the Planck constant. This can be calculated through the well known formula,

$$h = \left[1/2e^2\left(\mu_0/\epsilon_0\right)^{1/2}\right]/\alpha \tag{9}$$

where α is the fine structure constant. $\alpha = 7.297,353,08(10^-3)$ and is typically listed in physics reference tables as a unitless number. In fact as clearly demonstrated in the unpublished article by Jim Fisher, "Systems Analysis - Derivation of Essential Constants" this constant is actually the result of a relative value which has been imported into the system of absolute (Planck) scales. As such it has the units of $kg(m^2/s), absolute$.

The importance of this analogy is that while α is totally accepted, its origin and meaning are completely unknown. Further, since α is typically thought of as a unitless quantity, physics has not been really faced with having to assign it to any particular structure or particle property, and unfortunately likewise has had no means of making such an assignment if it was desired.

Thus understanding the formula for h, even though one of its factors α is unexplained, the formula for e is quite easy.

$$e = \left[\mu_0(G\epsilon_0)^{1/2}\right]A \tag{10}$$

where A is a geometric constant.

A is an interesting constant that can be calculated or decomposed as follows:

$$A = (2\pi)^{-3/2} \times (\pi\rho^2) = 1/2(2\pi)^{-1/2}\rho^2 \tag{11}$$

Here numerically ρ is a constant which was found to be:

$$\rho = 6/(6^2 + 1^2) \tag{12}$$

which has the generic geometric form of $a/(a^2+b^2)$ which becomes significant momentarily.

Using this numerical decomposition of A, we find, $A = 5.245,406,17(10^{-3})\,C^2m^{-1}$ in absolute Planck units. Using this value of A and the values of the three force constants, as listed in Table VIII in the Appendix, we find that with equation 10, we can calculate the value of e. The calculated value of $e_{calc} = 1.602,177,29(10^{-19})$ C as compared with $e_{measured} = 1.602,177,33(10^{-19}) \pm 4.9(10^{-26})$ C and finally the ratio of the measured to calculated is

$$\frac{e_{measured}}{e_{calc}} = 1.000,000,024 \tag{13}$$

The difference between the measured and calculated value of e is about 24 parts in 1 billion and well within the experimental error.

The derivation or origins of A are believed to be as follows. The constant ρ in the factor A represents the curvature κ, or the torsion τ of the cylindrical figure

$$\mathbf{R}(t) = aT_n^\dagger(Cos[F(t)])\mathbf{i}+aT_n^\dagger(Sin[F(t)])\mathbf{j}+bF(t)\mathbf{k} \tag{14}$$

for odd n. Here a = 6, b/n = 1, and F(t) is the trigonometrically substituted, cosine, Chebyshev $T_1^\dagger$, $T_3^\dagger$, and $T_5^\dagger$ polynomials for the electron, muon, and tau respectively. We will find that the 6 which occurs here as the amplitude coefficient of the planar **i** and **j** vectors also occurs in the planar radial equation that is part of the mass density or gravitational picture of the leptons.

Reviewing physics and engineering texts we find that Fourier decompositions-transforms-integrals are very frequently used when formalizing the discussions of wave patterns [28-29] In quantum mechanics texts in particular, such discussions center around probability densities in some non-consensus space, such as momentum space. There are a multitude of such examples in both theoretical and applied mathematical and engineering texts [28-29]. Thus the origin of the $(2\pi)^{-n/2}$ for n = 1 or 3, can probably be assigned to either a 1 or 3 dimensional Fourier manipulation of **R**(t) in charge space.

We can ask what does the implicit variable of time in the vector **R**(t) represent physically. If we return to an analogy with the photon, we see that this variable relates a measure of duration of events to position along the flight path of the particle. This same mathematical feature applied to the lepton would relate to events involving the circulation of the energy pattern around the center of the toroidal coil, the donut hole. This use of time or duration of events will become important, for its absence, in the discussion of the mass density equations describing the leptons.

While these assignments, of this $(2\pi)^{-3/2}$, and the $\pi\rho^2$ where $\rho = a/(a^2+b^2)$, a = 6 and b/n = 1 are not definitive, they are highly suggestive. While humans can physically experience velocity and acceleration, and can have an intellectual understanding of kinetic and potential energy as mathematical derivatives, they have not known the mathematical origin of their experience of charge. Here charge appears to be related to the square of the curvature or torsion of a generalized cylindrical wave expression which represents the electromagnetic structure of a particle. Curvature and torsion both involve first and second derivative expressions, the same as with the other mathematical expressions for energy. These expressions are invariant, as required. Finally, curvature and torsion, while although they are scalar quantities, are derived from vector expressions in space. This vector nature is in agreement with physics understanding of charge, and the electromagnetic fields and forces.

The equations used to derive the charge of the leptons involve exact analytical expressions. Uncertainty or the limits of accuracy is introduced through the physical constants used to scaleup these analytical expressions, from the world of mathematics to the consensus world of physics. Here the limiting constant is G, with certainty of only 3 decimals. This is not very satisfactory since e is measured to 6 decimals of certainty. The immediate impulse of physicists, as well of this author, is to rearrange the final equation

$e = F_1(G, \mu_0, \epsilon_0, geometry)$ to become

$G = F_2(e, \mu_0, \epsilon_0, geometry)$

While the logic of this work requires that G not depend on the leptons for its existence, this rearrangement is desirable in that G can now be calculated to much greater accuracy than that to which it can be measured. In the next immediate derivations for the masses of the leptons, a more accurate value for G is highly desirable. Accepting this improper but necessary logic, this calculated value of G is then used in the equations for the lepton masses through the overall scale factor of meters "absolute". Thus the origin of the calculated values of G and meters "absolute"shown in Table VIII in the Appendix.

It needs to be emphasized that the quantity $A = 5.245, 406, 17(10^{-3})$ $C^2 m^{-1}$used here and the quantities of mass / radial meter used in the next several sections are all referring to units in the absolute system of scales, "Planck units". As such these specific combinations of units have been demonstrated in the article "Systems Analysis" to be system independent. Thus they are not just mere many decimal accurate coincidences of the MKS relative system of units.

B. Masses of the Leptons

The general or generic form of the mass density equations developed for the leptons are shown below. The specific detailed applications of these equations for the electron, muon, and tau are given in subsections C, D, and E, respectively. The numerical results of using these equations are shown in Tables I through VI. Since this work is a mathematical endeavor, these results are presented so that the reader can reproduce, verify, and validate the "experimental" findings, if they so choose, before beginning any discussions as to their meaning. Likewise in the tables nine decimals are intentionally carried so that questions of computer calculation abilities and programming techniques can be settled. Table VI compares the ultimate calculated masses and mass energies of the leptons with their empirically measured values.

Beginning with the overall or final equation for calculating the mass of a lepton particle , m_p, we have:

$$m_p = C_g C_p D_p \tag{15}$$

where C_g is a general correlation constant or universal scaling constant.

$$C_g = e\mu_0(G\epsilon_0)^{1/2} = 4.893, 752, 96(10^{-36})\text{m abs} \tag{16}$$

C_p is the unitless individual particle constant; and D_p is the mass density function for the particle.

For the electron C_p is simply 1.0. For the higher members of the lepton series C_p is composed of three factors as follows:

$$C_p = F_c F_{mp} F_{sp} \tag{17}$$

F_c is a constant factor. F_{mp} is the series member factor for the particle. F_{sp} is a shielding or mass defect factor for the particle. The rational for these last two factors is discussed in Section VI.D.

$$F_c = \frac{1}{2\alpha} \tag{18}$$

$$F_{mp} = \left[\frac{a}{a^2 + b^2}\right]^2 \tag{19}$$

where $a = 6$ and $b = (n-1)^{1/2}$, and n is the number of the particle in the series. F_{sp} is best illustrated by the specific examples of the particles themselves and the discussions in Section VI.D.

Now for the important and crucial findings of this work, the mass density functions. The mass density function for the particles D_p can be determined by summing across the radial and angular mass density functions for each shell that a particle may have.

$$D_p = \sum_{k=1}^{n} S_{pk} D_{pk}(r) D_{pk}(\theta) \tag{20}$$

where all three factors S_{pk}, $D_{pk}(r)$, and $D_{pk}(\theta)$ are specific to the particle, p and the shell, k, that one is calculating. S_{pk} is a unitless shell correlation constant illustrated with the specific particles and discussed in Section VI.D. $D_{pk}(r)$ is the radial mass density function and $D_{pk}(\theta)$ is the angular mass density function. As we will soon see in the specific examples of the leptons all three of these factors form unique patterns or mathematical series. The specific features of the radial and angular functions are discussed in detail in Section VI. Graphical presentations of both the radial and angular functions for each lepton can be seen in the figures at the end of the article

One of the new features of this work is the embedded temporal parameters. These are different from the strictly spatial parameters found with the wave equations describing the electron shells of the hydrogen atom. Here embedded within both the radial and angular spatial mass density functions we have radial and angular temporal mass density functions.

We find in the radial expressions for all the leptons that there are two temporal factors of the implicit variable $r(t_r)$, an exponential and a polynomial. Here t_r

means radial time. This function is the same for all the leptons. It represents the distance along the parabolic curve $(a\pi t_r^2)$. This curve describes the area as we progress outward across a uniform "gray" disk, the mature radial energy condition in time. Note this is the instantaneous distance, not cumulative, along the curve, not to the curve. Specifically

$$r(t_r) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} ds \left(\frac{2\pi t_r^2}{k^{1/2}}\right) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[1 + \left(\frac{4\pi t_r}{k^{1/2}}\right)^2\right]^{\frac{1}{2}} dt \tag{21}$$

where k is the Fraunhofer Diffraction Constant as defined immediately below.

Another unique feature of this work, in the realm of particle physics, is the occurrence of initial conditions for both the radial and angular mass density equations. In applied engineering texts we find initial conditions are a common feature used to make the solutions to second order differential equations specific to the application being discussed. Thus initial conditions might be expected here where both the radial and angular mass density equations appear to be the solutions to second order differential equations, which are both time and space dependent.

In the radial expression for all the leptons, the initial condition in radial time is the same.

$$I(r) = F_{dfn}[F(r)] = \left[\frac{2J_1[F(r)]}{F(r)}\right]^2 \tag{22}$$

Where $F_{dfn}[F(r)]$ is the Fraunhofer Diffraction Function and J_1 is the first order Bessel Function of the first Kind. Specifically $F(r) = kr$ where

$$k = 1.697, 525, 53... = \int_0^\infty F_{dfn}(1.000, 000, ...)r^1 dr \tag{23}$$

Note that $\int_0^\infty F_{dfn}(kr^1)dr = 1.000, 000, ...$ and thus is a self normalizing initial distribution.

See Born and Wolf [30] for the origins of the Fraunhofer Diffraction Function in its classical historic setting. There in Chapter 8, the details of this mathematical form are rigorously derived in terms of the parameters; k - the wave number, a - the aperture radius, and w - the radius of discussion across the pattern. For this work these parameters have been rolled together to become the monomial $F(r)$.

An initial or normalizing constant for the radial equations is needed.

$$C_{rpk} = \int_0^\infty I(r)D_{pk}(r)dr \tag{24}$$

The angular mass density functions need further minor explanation before we get to the specific use with the leptons themselves. For the leptons there is only one angular spatial dimension, unlike the two angular spatial dimensions found with the hydrogenic electron shells. The integrated expression for each shell of each particle is multiplied by a common angular multiplying factor, the number 4, which is a composite. It is a product of a multiplier of 1/2 outside the Chebyshev polynomial, of 2 for angular symmetry of the integral about zero, and of 2 orthogonal forms being simultaneously applicable. This angular function is then repeated within the integral of the initial angular condition times the angular function, giving an overall common multiplier of 4.

The two orthogonal angular forms have the general form

$$F(\theta) = T_n^\dagger(Sin[\pi/2 \ \ \theta(t_\theta)]) \tag{25}$$

where $\theta(t_\theta) = T_n^\dagger[Sin(n^{-1}t_\theta)]$ or $= T_n^\dagger[Cos(n^{-1}t_\theta)]$ where t_θ means angular time.

In the angular expression for all the leptons, the initial condition in the angular time is the same.

$$I(\theta) = cos(\theta) \tag{26}$$

Note that $\int_0^{\pi/2} cos(\theta)d\theta = 1$ and thus is a self normalized initial distribution in space. Thus it is also a self normalized distribution as used as the argument for the interior or implicit temporal $T_n^\dagger$ polynomial. The initial constants in the angular equations are calculated as.

$$C_{\theta pk} = \int_0^{\pi/2} I(\theta)D_{pk}(\theta)d\theta \tag{27}$$

Finally, the appropriate normalizing factors for the Laguerre and Chebyshev T orthogonal polynomials are always used, except in the calculation of the initial radial and angular constants. These need to be remembered since they have been suppressed in all the equations above and those which follow. This was done so as to maintain the clarity and focus of the primary form and appearances of these equations. The effect of these normalizing factors are included in the tables.

C. Electron specific equations and calculations

We finally get to the pay off of the equations of ultimate interest. We obtain the following equations for the electron,(p = 1 or e) in the equations above. The electron consists of only one shell since the L_0 polynomial has no derivatives. Due to the electron being the first member of the series the equations for its mass density are quite simple.

$$D_{11}(r) = C_{r11} \int_0^\infty e^{-6r^2} e^{r(t_r)} L_0^0(r(t_r))dr \tag{28}$$

and

$$D_{11}(\theta) = C_{\theta 11} \int_0^{\pi/2} T_1^\dagger(Sin[\pi/2 \ \ \theta(t_\theta)])d\theta \tag{29}$$

where $C_e = 1$ and $S_1 = 1$. Therefore the combined radial-angular mass density function for the electron becomes:

$$m_e = C_g C_e \sum_{k=1}^{1} S_k D_{1k}(r) D_{1k}(\theta) \tag{30}$$

These equations were used to calculate the values shown for the electron in Tables I through VI. As seen the results show that the ultimate calculated mass-energy is within 3 parts in 10 million to the experimentally determined value.

D. Muon specific equations and calculations

We obtain the following equations for the muon,(p = 2 or μ) in the equations above.The muon has two shells, a primary represented by the L_2^0 polynomial and a secondary by the only even derivative of L_2, the L_2^2 polynomial.

$$D_{21}(r) = C_{r21} \int_0^\infty e^{-6r^2} e^{r(t_r)} L_2^0(r(t_r))dr \tag{31}$$

$$D_{22}(r) = C_{r22} \int_0^\infty e^{-6r^2} e^{r(t_r)} L_2^2(r(t_r))dr \tag{32}$$

$$D_{21}(\theta) = 4C_{\theta 21} \int_0^{\pi/2} T_3^\dagger(sin[\pi/2 \;\; \theta(t_\theta)])d\theta \tag{33}$$

$$D_{22}(\theta) = 1/3D_{11}(\theta) \tag{34}$$

The overall particle constant C_μ can be determined as follows.

$$C_\mu = F_c F_{m\mu} F_{s\mu} \tag{35}$$

as shown in 17 above. From the simple algebra of 18 and 19 we find $F_c = 6.851, 799, 475(10^1)$ and $F_{m\mu} = 2.629, 656, 683(10^-2)$ respectively. $F_{s\mu}$ the shielding or mass defect factor is;

$$F_{s\mu} = \frac{1}{2}\left[\frac{1}{1 - 1/3!}\right] \tag{36}$$

The individualizing shell factors $S_1 = 1$ and $S_2 = 1 + 1/2 \times 1/37$. Therefore the mass of the muon can be calculated as

$$m_\mu = C_g C_\mu \sum_{k=1}^{2} S_k D_{2k}(r) D_{2k}(\theta) \tag{37}$$

These equations were used to calculate the values shown for the muon in Tables I through VI. As seen the results show that the ultimate calculated mass-energy is within 1 parts in 10 million to the experimentally determined value.

E. Tau specific equations and calculations

Similarly, we can determine the equations for the tau particle (p = 3 or τ). As seen the tau has three shells since the L_4 polynomial has a base L_4^0 and two even derivatives, L_4^2 and L_4^4.

$$D_{31}(r) = C_{r31} \int_0^\infty e^{-6r^2} e^{r(t_r)} L_4^0(r(t_r))dr \tag{38}$$

$$D_{32}(r) = C_{r32} \int_0^\infty e^{-6r^2} e^{r(t_r)} L_4^2(r(t_r))dr \tag{39}$$

$$D_{33}(r) = C_{r33} \int_0^\infty e^{6r^2} e^{r(t_r)} L_4^4(r(t_r))dr \tag{40}$$

$$D_{31}(\theta) = 4C_{\theta 31} \int_0^{\pi/2} T_5^\dagger(Sin[\pi/2 \;\; \theta(t_\theta)])d\theta \tag{41}$$

$$D_{32}(\theta) = 1/5D_{21}(\theta) \tag{42}$$

$$D_{33}(\theta) = 1/5D_{11}(\theta) \tag{43}$$

$$C_\tau = F_c F_{m\tau} F_{s\tau} \tag{44}$$

where F_c is unchanged and equal $6.851, 799, 475(10^1)$. Again using the simple algebra of 19 we find $F_{m\tau} = 2.493, 074, 792(10^-2)$. $F_{s\tau}$ the shielding or mass defect factor is;

$$F_{s\tau} = \frac{1}{4}\left[\frac{1}{1 - 1/3! - 2^{9/4}/5!}\right] \tag{45}$$

The shell factors S_1 and S_2 are unchanged and $S_3 = 1 + 1/4 \times 1/37$. Thus putting all the pieces together the mass of tau becomes

$$m_\tau = C_g C_\tau \sum_{k=1}^{3} S_k D_{3k}(r) D_{3k}(\theta) \tag{46}$$

These equations were used to calculate the values shown for the tau in Tables I through VI. As seen the results show that the ultimate calculated mass-energy is within 2 parts in 10 thousand to the experimentally determined value.

V. A POSSIBLE 4TH MEMBER OF THE LEPTON FAMILY?

On seeing the general pattern of these mass density equations, one should ask what happens for the higher even member Ln polynomials, those above L_4 of the τ? On checking these, we will find that the curve of increasing mass density with the row number of the L_n polynomials curls over and rapidly goes negative. The 4th member of this series, the row L_6 and its derivatives, was calculated using the equations below and is mathematically possible.

Following the general equations, the specific equations for this 4th member are as follows.

$$D_{41}(r) = C_{r41} \int_0^\infty e^{-6r^2} e^{r(t_r)} L_6^0(r(t_r))dr \quad (47)$$

$$D_{42}(r) = C_{r42} \int_0^\infty e^{-6r^2} e^{r(t_r)} L_6^2(r(t_r))dr \quad (48)$$

$$D_{43}(r) = C_{r43} \int_0^\infty e^{6r^2} e^{r(t_r)} L_6^4(r(t_r))dr \quad (49)$$

$$D_{44}(r) = C_{r44} \int_0^\infty e^{6r^2} e^{r(t_r)} L_6^6(r(t_r))dr \quad (50)$$

$$D_{41}(\theta) = C_{\theta 41} \int_0^{\pi/2} T_7^\dagger(Sin[\pi/2 \;\; \theta(t_\theta)])d\theta \quad (51)$$

$$D_{42}(\theta) = 1/7 D_{31}(\theta) \quad (52)$$

$$D_{43}(\theta) = 1/7 D_{21}(\theta) \quad (53)$$

$$D_{44}(\theta) = 1/7 D_{11}(\theta) \quad (54)$$

$$C_{4th} = F_c F_{m4th} F_{s4th} \quad (55)$$

where again the constant F_c is as above, and 19 gives $F_{m4th} = 2.366, 863, 9(10^-2)$. F_{s4th} the shielding or mass defect factor is;

$$F_{s4th} = \frac{1}{8} \text{or} \frac{1}{6} \left[\frac{1}{1 - 1/3! - 2^{9/4}/5! - 4^{16/9}/7!} \right] \quad (56)$$

and $S_4 = 1$ approximately. Therefore the mass of the fourth member becomes

$$m_{4th} = C_g C_{4th} \sum_{k=1}^{4} S_k D_{4k}(r) D_{4k}(\theta) \quad (57)$$

Tables I through V show the numerical values for this fourth member of the Lepton family.

As seen the density of this hypothetical particle **lies somewhere between that of the electron and that of the muon**. The exact value of this mass density can not be predicted since a strong pattern has not been set for several of the particle scale factors.

If we draw radial-angular plots, polar coordinates, for the angular equations of the muon, tau, and this 4th member, then we find a thumb amongst several fingers. The plots of both the angular equations and those of the angular equations multiplied by the initial condition all show a clearly imbalanced lobe amongst the other lobes of the plots. See Figures 1-4 at the end of the article. This imbalance gets accentuated the higher we go in the series. This out-of-balance angular appearance is probably directly related to the decreasing half life between the muon and tau. Although this 4th member is not rigorously excluded mathematically, the gross imbalance of its angular appearance combined with a more complicated radial equation that stabilizes less energy than the two previous simpler members, muon and tau, probably explains why this particle has never been observed.

Additionally, machines on which older low energy collider data was collected, may not have been able to produce a fine enough scattering matrix to indicate that some of the collision products were the result of not only a low energy intermediate, but also an extremely short lived particle only able to travel a short distance.

Aside from the machinery, there is the human element. The known lepton and quark series set up an appearance which could lead to logical trap for the particle physicists studying these series. First these elementary particle series give the appearance of always increasing in mass as we progress up through the series. Secondly, higher energy for an elementary particle, quark or lepton, always appears to go hand-in-hand with a shorter half life. This may be true, but it sets up invalid logic. High energy yields short half life, therefore short half life must always come from high energy particles.

VI. DISCUSSION AND CONCLUSIONS

A. The Mass Density Equation Parameters and the Charge Equation Parameter

In the mass density equations for the leptons we found two embedded or implicit temporal parameters. The distinct appearances of radial time t_r and angular time t_θ were intentional. There appears to be no requirement that these two variables be the same. Thus the conceptual possibility was left open. Further, if these equations are to represent the solutions to some time dependent Schrodinger style wave equation after the separation of variables, then it is mandatory that these two expressions of t be independent. Otherwise the radial and angular parameters would be linked and the solutions to the equations would not be separable.

Besides these two variables of time, there is the third temporal parameter. This parameter t was found in the vector expressions shown in 14 which lead to the equation for the charge of the leptons. It appears to be independent of the two variables t_r and t_θ found in the mass density equations. This parameter measures events relating to the circulation of the energy pattern around the center of the donut. While we might logically expect that one cycle length or revolution around the donut would coincide with one rotation or spin of the periphery of the radial planar figure about its center, these equations give no mathematical guarantee of this. Likewise, there is no requirement that the t of $\mathbf{R}$(t) and t_θ of $D_{pk}(\theta)$ be integer multiples of one another, or have any relation at

TABLE I: **Values of Lepton Radial Equation Integrals**

Particle Shell	Initial Constant C_{rpk}	Integrated Equation $D_{pk}(r)$ w/o C_{rpk}	Product $D_{pk}(r)$
Electron			
Primary	$1.618,533,691(10^2)$	$3.428,165,302(10^2)$	$5.548,601,040(10^4)$
Muon			
Primary	$6.760,706,674(10^3)$	$1.943,599,062(10^4)$	$1.314,010,315(10^8)$
Secondary	$1.618,533,691(10^2)$	$2.424,078,932(10^2)$	$3.923,453,422(10^4)$
Tau			
Primary	$2.387,176,656(10^4)$	$1.089,901,363(10^5)$	$2.601,787,092(10^9)$
Secondary	$4.116,467,332(10^3)$	$3.699,379,637(10^3)$	$1.522,837,542(10^7)$
Tertiary	$1.618,533,691(10^2)$	$6.997,713,119(10^1)$	$1.132,603,445(10^4)$
4th Member			
Primary	$1.179,803,559(10^2)$	$6.949,401,001(10^4)$	$8.198,928,031(10^6)$
Secondary	$5.557,260,386(10^3)$	$6.878,138,841(10^3)$	$3.822,360,851(10^7)$
Tertiary	$-7.069,341,479(10^3)$	$3.987,278,720(10^2)$	$-2.818,743,484(10^6)$
Quaternary	$1.618,533,691(10^2)$	$1.277,601,775(10^1)$	$2.067,841,518(10^3)$

TABLE II: **Values of Lepton Angular Equation Integrals**

Particle Shell	Initial Constant $C_{\theta pk}$	Integrated Equation $D_{pk}(\theta)$ w/o $C_{\theta pk}$	Symmetry	Product $D_{pk}(\theta)$
Electron				
Primary	$0.890,365,284$	$0.941,966,611$	4	$3.354,777,477$
Muon				
Primary	$0.442,427,296$	$0.152,908,897$	4	$0.270,604,279$
Secondary	$0.890,365,284$	$0.313,988,870$	4	$1.118259,159$
Tau				
Primary	$0.436,375,136$	$0.264,779,514$	4	$0.462,172,786$
Secondary	$0.442,427,296$	$0.030,581,779$	4	$0.054,120,856$
Tertiary	$0.890,365,284$	$0.188,393,322$	4	$0.670,955,495$
4th Member				
Primary	$0.276,612,505$	$0.138,706,718$	4	$0.153,472,051$
Secondary	$0.436,375,136$	$0.037,825,645$	4	$0.066,024,684$
Tertiary	$0.442,427,296$	$0.021,844,128$	4	$0.038,657,754$
Quaternary	$0.890,365,284$	$0.134,566,659$	4	$0.479,253,925$

all.

We found in 10 concerning the calculation of the charge the geometric parameter A with units of C^2/m_{abs}. We can ask in this setting what is the meaning of this basic human measuring stick, meters? In this setting of a rectilinear vector expression, meters are a measure of distance perpendicular to the electromagnetic surface being discussed.

In the scalar radial-angular mass density equations we found that the grand total expression in 15 up to the point that the general correlation constant, C_g, is applied that the expressions result in mass density units of kg/m_{abs}. There meters can be thought of as a measure of distance co-linear, parallel, or simultaneous with the stabilized gravitational force or energy pattern being discussed.

Aside from the basic spatial and temporal measuring devices, there are the parameters of C^2 and kg. These two of course are the objectives of the calculational framework. These represent different measurements or descriptions of the encapsulated or stabilized energy of the particles. Even here we can see a very simple pattern. Coulombs representing the binary force electromagnetic are described by a "2 dimensional" phenomena the curvature and are squared. Kilograms representing the unary force gravity is first order. Likewise this measure is described by a simple linear or radial phenomena, that just happens to tumble around in multiple dimensions with

TABLE III: **Radial - Angular Products**

Particle Shell	Radial-Angular Product $D_{pk}(r) \times D_{pk}(\theta)$	Shell Factor S_{pk}	Final Shell Contribution D_{pk}
Electron			
Primary	$1.861,432,180(10^5)$	1	$1.861,432,180(10^5)$
Sum of Shells			$1.861,432,180(10^5)$
Muon			
Primary	$3.555,768,139(10^7)$	1	$3.555,768,139(10^7)$
Secondary	$4.387,437,724(10^4)$	$1.013,513,514$	$4.446,727,423(10^4)$
Sum of Shells			$3.560,214,867(10^7)$
Tau			
Primary	$1.202,475,190(10^9)$	1	$1.202,475,190(10^9)$
Secondary	$8.241,727,105(10^5)$	$1.013,513,514$	$8.353,101,796(10^5)$
Tertiary	$7.599,265,053(10^3)$	$1.006,756,757$	$7.650,611,438(10^3)$
Sum of Shells			$1.203,318,151(10^9)$
4th Member			
Primary	$1.258,306,297(10^6)$	1	$1.258,306,297(10^6)$
Secondary	$2.523,701,665(10^6)$	$1.013,513,514$	$2.557,805,741(10^6)$
Tertiary	$-1.089,662,926(10^5)$	$1.006,756,757$	$-1.097,025,514(10^5)$
Quaternary	$9.910,211,643(10^2)$	$1.004,504,505$	$9.954,852,236(10^2)$
Sum of Shells			$3.707,404,972(10^6)$

TABLE IV: **Particle Scale Factors of Leptons**

	Common Constant C_g, m_{abs}	Constant Factor F_c	Member Factor F_{mp}	Shielding Factor F_{sp}	Particle Constant C_p	Product $C_g \times C_p$, m_{abs}
Electron	$4.893,752,96(10^{-36})$		1	1	$1.000,000,000$	$4.893,752,96(10^{-36})$
Muon	$4.893,752,96(10^{-36})$	$68.517,994,746$	$0.026,296,567$	$0.600,000,000$	$1.081,072,817$	$5.290,503,30(10^{-36})$
Tau	$4.893,752,96(10^{-36})$	$68.517,994,746$	$0.024,930,748$	$0.314,983,211$	$0.538,055,850$	$2.633,112,41(10^{-36})$
4th Member	$4.893,752,96(10^{-36})$	$68.517,994,746$	$0.023,668,639$	$0.157,955,887$	$0.256,161,435$	$1.253,590,78(10^{-36})$

time.

B. Discussion of Angular Equations

Plots of the angular equations of the leptons, both radial-angular and rectilinear presentations, can be found at the end of the article. Figures 1-2 show the appearance of the angular equations and Figures 3-4 show the appearance of the angular equations multiplied by the initial condition.

The angular equations of the leptons, both spatial and temporal, are described by the odd membered trigonometrically substituted Chebyshev $T_n^\dagger$ orthogonal polynomials. The reasoning for this is as follows.

First we assume a second order differential equation description for a series of stable but otherwise unknown energy patterns, such as the leptons. Next we assume the validity of the six major assumptions which permit the separation of the variables representing the various spatial dimensions. Proceeding through the complete separation of variables, we find the following.

For angle 1 the equation is:

$$Cos^{-0}(\theta_1)\frac{d}{d\theta_1}\left[Cos^{0}(\theta_1)\frac{dH_2}{d\theta_1}\right] - (-qn_2)H_2(\theta_1) = 0 \tag{58}$$

For angle k, where $1 < k < (n-1)$

$$Cos^{-(k-1)}(\theta_k)\frac{d}{d\theta_k}\left[Cos^{k-1}(\theta_k)\frac{dH_{k+1}}{d\theta_k}\right] - (-qn_{k+1}Cos^{-2}(\theta_k) - qn_{k+1})H_{k+1}(\theta_k) = 0 \tag{59}$$

and for angle n-1, the equation becomes:

$$Cos^{-(n-2)}(\theta_{n-1})\frac{d}{d\theta_{n-1}}\left[Cos^{n-2}(\theta_{n-1})\frac{dH_n}{d\theta_{n-1}}\right] - (-qn_{n-1}Cos^{-2}(\theta_{n-1}) - qn_1)H_n(\theta_{k-1}) = 0 \tag{60}$$

Here H_{k+1} is an arbitrary function of the θ_k argument, and qn_k is an arbitrary constant or quantum number.

TABLE V: **Results of Derivations for Mass of Leptons**

	Radial Angular Product D_p, kg/m abs	Scale Multiplier $C_g \times C_p$, m abs	Calc Mass m_p, kg
Electron	$1.861,432,180(10^{5})$	$4.893,752,96(10^{-36})$	$9.109,389,239(10^{-31})$
Muon	$3.560,214,867(10^{7})$	$5.290,503,30(10^{-36})$	$1.883,532,849(10^{-28})$
Tau	$1.203,318,151(10^{9})$	$2.633,112,41(10^{-36})$	$3.168,471,956(10^{-27})$
4th Member	$3.707,404,972(10^{6})$	$1.253,590,78(10^{-36})$	$4.647,568,702(10^{-30})$

TABLE VI: **Comparison of Lepton Mass Derivations to Measurements**

	Measured Mass-Energy	Measured / Calculated
Electron	MeV/c^2	
High	$5.109,992,1(10^{-1})$	1.000,000,3
Mid	$5.109,990,6(10^{-1})$	1.000,000,0
Low	$5.109,989,1(10^{-1})$	1.000,000,3
Muon		
High	$1.056,584,23(10^{2})$	1.000,000,24
Mid	$1.056,583,89(10^{2})$	0.999,999,91
Low	$1.056,583,55(10^{2})$	0.999,999,59
Tau		
High	$1.777,34(10^{3})$	0.999,98
Mid	$1.777,05(10^{3})$	0.999,81
Low	$1.776,79(10^{3})$	0.999,67

Here the first dimension is assumed to be the radial parameter, and has H_1 and the other appearance of qn_1 associated with it. We assume some $F(x_k) = H_{k+1}(\theta_k)$ for each of the n-1 separated angular equations, and assume $x_k = f(\theta_k) = Sin(\theta_k)$. Taking the required derivatives and substituting the results into the first term of the n-1 angular equations, we find this first term appearances becomes as in Table VII. Considering the differential equation formulations for the orthogonal polynomials, we find the correspondences as shown.

A definitive discussion of all these orthogonal polynomials can be found in chapter 22 of Abramowitz and Stegun [31].

Thus we find a systematic orderly progression of correspondences between the differential equations for the orthogonal polynomials and the first term of the trigonometrically substituted angular equations which could result from some stable n-dimensional energy pattern.

The quantum mechanic description of the hydrogenic electron orbital shells follows this progression for the third spatial dimension of discussion. The Legendre P_n polynomials, technically the Jacobi $P_n[0, 0, Cos(\theta_2)]$ polynomials and their derivatives, are used successfully to describe the geometric appearance of the spherical or second angular dimension. For the planar mathematics or first angular dimension, though, the Mass Density description assumed for the hydrogenic orbitals deviates and is described as

$$D(\theta_1), \text{hydrogenic electron orbitals} = qn_2 e^{-i\sqrt{qn_2\theta_1}} \tag{61}$$

where $i = \sqrt{-1}$.

Here with the leptons for the first and only angular parameters, for both the temporal and spatial angles, we simply stay with the logical pattern of using the trigonometrically substituted Chebyshev $T_n^\dagger$ polynomials. As seen in Tables II through VI, this assumed description gives the desired results.

Referring to the generalized cylindrical figure of equation 2 we found that to produce the constant value of the electrical charge of the leptons that G(t) needed to equal F(t). Applying this to the angular Mass Density equations for the leptons we end up with the curious appearance of

$$D(\theta)\text{for the leptons} = T_n^\dagger(Sin[\pi/2\theta(t_\theta)]) \tag{62}$$

where the embedded $\theta(t_\theta)$ is itself $= T_n^\dagger[Sin(n^{-1}t_\theta)]$.

The $\pi/2$ is necessary so that the exterior $T_n^\dagger$ polynomial covers its full range of 0 to +1. The n-1 assures that the outside spatial Sin function covers a range of π over the integration. We need to remember that the integrals for the angular expressions of the leptons, both those for the initial constant and for the angular equation itself, are integrals of substituted rectilinear orthogonal polynomials. These represent the solutions to some unspecified second order differential equations. Thus the various angular equations are integrated from $-\pi/2$ to $\pi/2$, or $2 \times (0 \text{ to } \pi/2)$, values which correspond to the valid range of the substituted original orthogonal polynomials. These are not polar or spherical mathematical forms. Thus the simple rectilinear integrals $d\theta$ are used, and not the form $(rdrd\theta)$ used to find polar areas.

We find that the two appearances $Sin[aCos(bt_\theta)]$and $Sin[aSin(bt_\theta)]$work equally well, are orthogonal to each other, and just phase shifted or represent two possible orientations of the angular function θ about the arbitrary starting point of the θ polar line. The implicit trigonometrics of $Sin(t_\theta)$ and $Cos(t_\theta)$ result in stable cyclic and bounded figures, unlike the open ended λt which results in the unbounded photon. The embedded function of angular time t_θ produces an interesting concept. Thinking of the t_θ polar line as the present, proceeding angularly clockwise away from the t_θ polar line can be viewed as

TABLE VII: General Angular Equation Correspondences

Angle	Equation 1st Term Appearance	ODE Orthogonal Polynomials
1	$(1-x^2)\ d^2F/dx^2 - 1x\ dF/dx$	Chebyshev $T_n(x)$
2	$(1-x^2)\ d^2F/dx^2 - 2x\ dF/dx$	Jacobi $P_n(a,b,x), a=b=0$
3	$(1-x^2)\ d^2F/dx^2 - 3x\ dF/dx$	Ultraspherical $C_n(a,x), a=1$
k even	$(1-x^2)\ d^2F/dx^2 - kx\ dF/dx$	Jacobi $P_n(a,b,x); a=b=(k-2)/2$
k odd	$(1-x^2)\ d^2F/dx^2 - kx\ dF/dx$	Ultraspherical $C_n(a,x); a=(k-1)/2$

going into the past, and anticlockwise as going into the future.

The secondary and tertiary shells of the leptons are not described by the derivatives of the $T_n^\dagger$ polynomials. Reviewing the higher hydrogenic electron shells, the derivatives of the $P_n(\theta_2)$ polynomials give correct mathematical descriptions. This is because a mathematical "trick" can be employed to maintain the orthogonality of the derivatives of the Pn polynomials. The derivatives of the P_n polynomials can be multiplied by $(1-x^2)^{derivative}$ $^{order/2}$ to force them to comply with the defining differential equations for the original functions and to simultaneously still maintain their orthogonality. There appears to be no similar trick which can be used to modify either the form or the weight factors of the derivatives of the $T_n^\dagger$ polynomials.

The initial condition of $Cos(\theta)$ probably represents the linear or one dimensional energy pattern of a particle-wave running unimpeded around a smooth circular ring. The angular equations of the upper lepton members then mature into flower petal-like appearances as mature functions in time. We are fortunate that this simple initial angular condition applies to all the members of the series, and is a self normalized distribution.

C. Discussion of Radial Equations

Plots of the radial equations of the leptons can be found at the end of the article. Figure 5 shows both the un-scaled and the final appearances of these equations.

The radial mass density equations of the leptons have the generic form

$$D(r) = e^{-ar^2} e^{r(t_r)}[\text{polynomial expression in } r(t_r)] \quad (63)$$

where $r(t_r)$ is a function of radial time.

There are three factors here;

1. e^{-ar^2} This is the attenuator or longevity factor. This factor will overpower all other factors, of exponential order, and ultimately terminate the expression or bring it to converge to some value.

2. $e^{+r(t_r)}$ This is the driver and represents the real force, intensity, or effort that sustains the particle.

3. polynomial expression in $r(t_r)$ This is the shape factor. It gives form, shape, or direction to the effort of the second factor.

Viewed in this manner we can easily see how all three factors are necessary. Both the primary spatial function e^{-ar^2} and the primary temporal function $e^{+r(t_r)}$ involve exponentials and thus could easily derive from or evolve into differential equations, of either the first or second order.

There is an interesting aspect of the initial radial mass density distribution seen in the Fraunhofer Diffraction Function of Equation 22. When it is used as a radial function in energy calculations and when $F(r) = ar^1$, then it effectively incorporates a modified version of the inverse square law with the factor $F(r)^2$ in the denominator. We can be assured that the initial radial mass density Fraunhofer diffraction pattern is not really the result of diffraction, but rather represents some energy pattern whose mathematics are incidentally identical to that of Fraunhofer diffraction. This initial condition probably represents the two dimensional radial energy pattern resulting from a particle in a flat circular box. As with the angular initial condition, this radial initial condition is self-normalized. Further mathematical gratuity is the same initial condition applies to all the members of the series, as well as to all the shells of the upper members of the series.

Examination of the ultimate variables of $D(r)$ reveal an interesting property. R in space and t_r in radial time are always to the second power. Although the outside appearance and behavior of the radial function of time (Equation 21) is that of a pseudo 1st order, internally t_r is squared. Thus we can think of the radial functions as symmetric in space and time, with negative values of t_r extending inward into the past and positive values extending outward into the future. The only conceptual versatility we need is that in visualizing radial-polar plots which extends inward to negative values of r or t_r. Typical radial-angular plots stop with r = 0, or t_r in this case, as a dot at the origin. We only need to expand this origin outward into a circle of r or $t_r = 0$, with inside the circle having negative r or t_r, and outside the circle having positive r or t_r. Physically this means the leptons are in a parabolic energy well in radial time, with the present at the bottom of the well.

When viewing the overall pattern of the radial equations across all the members of the series we are struck by the even-ness of this series. The members occur only at even values of n for the Laguerre L_n polynomials. We find that odd n produces negative values for the radial integrals. Additionally the auxiliary shells for the upper members only occur at even numbered derivatives. Looking back at the angular equations we find that the members of the series only occur at the odd T_n polynomials. Thus neither the radial equations nor the angular equations occur by a continuous sequential polynomial series.

One major outstanding uncertainty is the exact relationship between the 6 occurring here as the coefficient of r^2 in the primary exponential, e^{-6r^2}, and the 6 occurring as the amplitude coefficient of the planar **i** and **j** vectors of the toroidal coil which gives meaning to the charge equation.

D. Discussion of Scale Factors

Aside from the three factors (driving, shaping, and attenuating) of the radial equation, a fourth factor is necessary for a real particle to come into being. A factor is needed which gives concreteness or materialization to the mathematical expressions. This is the scale factor which mathematically translates from the arbitrary scale of math-geometry to the scale of the consensus world of humans. Here is where the real world intrudes upon what to this point has been purely sterile mathematic-geometric equations. This factor, typically a premultiplier external to any exponentials, trigonometrics, et cetera, is composed of physics constants. A typical examples of this might be the $8m(\pi/h)$ found in the Schrodinger wave equation for the electron shells of the hydrogen atom, or the conversion factor G found in $F = Gm_1m_2/r^2$. It is this correlation constant which turns what otherwise would remain a correlation into an actual equation. Thus at least one general scale factor must be involved in these equations. The math-geometric portion of the mass density equations results in units of, kg / m, and needs to be multiplied by a quantity, meters. As discussed with the charge of the leptons the value of C_g (see Equation 16) needed the accuracy of G improved by back calculation from the equation for the charge.

One finds as they step through the periodic chart, the best known mathematical-physical series, that each member has some uniqueness, some specific quirks of their own. We can not predict the details of each member of the whole periodic chart by just examining the first element, hydrogen. Likewise, here the mass density equations for the first lepton member, the electron, are so simple that we can logically expect some added complications to arise when moving to the higher members of the series. The first factor contributing to the individual particle's uniqueness has been called the series member factor previously. This factor has the generic form

$$F_k = \left[\frac{1}{2\alpha}\right]\left[\frac{a}{a^2+b^2}\right]^2 \tag{64}$$

where $a = 6$, and $b = (k-1)^{1/2}$.

This again has the appearance of ρ^2 found in the equation describing the charge of the leptons, and ultimately has the form of the curvature or torsion of a generalized cylindrical spiral, or a toroidal coil in this case.

The second factor contributing to the particle's uniqueness has been called the shielding factor. This modification or mathematical factor appears to describe some sort of "shielding", "binding energy", or "mass defect" in going from the electron to the muon to the tau. This factor appears to have the form

$$F_\mu = 1/2[1 - 1^{4/1}/3!]^{-1} \tag{65}$$

$$F_\tau = 1/4[1 - 1^{4/1}/3! - 2^{9/4}/5!]^{-1} \tag{66}$$

where the general form appears as follows:

$$F_p = 1/2^{n-1}[1 - 1^{4/1}/3! - 2^{9/4}/5! - 4^{16/9}/7! - ...]^{-1} \tag{67}$$

Unfortunately since the muon and tau are only two members of a series, and since the mass of the tau has not been measured to the accuracy of that of the electron and muon, the pattern is not well established.

One more individualizing factor was discernible, that which gave uniqueness to the individual shells of the higher members of the series. For the muon with a 7 decimal energy measurement and the contribution of its secondary shell only 3 orders of magnitude smaller than that of its primary shell, this factor is absolutely necessary and is mathematically precise. The mathematical accuracy and simplicity of this factor tend to preclude it from being a coincidence. The form found for this multiplication factor for the secondary shell of the muon is

$$S_{\mu 2} = \left[1 + \left(\frac{1}{2}\right)\left(\frac{1}{37}\right)\right] \tag{68}$$

Little effort is needed to recognize this $1/37$ as $1/(6^2 + 1^2)$. For the tau with only 5 decimals of measurement accuracy and with its secondary shell contributing 4 orders of magnitude less energy than its primary, the effects of this factor are not discernible. Similarly, for the tertiary shell of the tau this individualizing factor can not be specified.

E. Conclusions

The specific objective of this work was; starting with the three universal force constants (G, μ_0, ϵ_0) as logically a-priori and to develop mathematical equations which explain some of the fundamental measured physical properties of the leptons. Equations were found which predict or match the measured charge of the leptons, and the measured masses of the three leptons to the required accuracy.

The nature of these equations is as follows. In general form, the mass density equations are similar to those describing the electron shells around the hydrogen atom.

They contain a planar radial equation in space and one angular equation in space. These spatial equations in turn both contain embedded or implicit temporal equations, and each have initial conditions in time that result in multiplying factors. The radial equations have two exponentials multiplying an appropriate member of the Laguerre orthogonal polynomial series. The angular equations, both the external spatial equations and the internal temporal equations, are trigonometrically substituted Chebyshev $T_n^\dagger$ orthogonal polynomials. There are also a series of well defined factors which serve to scale from an arbitrarily sized realm of math-geometry to the consensus world of physics.

The general geometric appearance found for these particles was that of a toroidal coil. The equation for the charge of the leptons follows quite simply and directly from the vector formulation of the curvature or torsion of this toroidal description.

This mathematical geometric framework leads to several additional conclusions.

The leptons, and thus logically the neutrinos and quarks, have definable structures and are not mathematical points. Albeit, the diameters of these structures or wave patterns probably are many orders of magnitude too small for physicists to measure.

Time is two dimensional, at least. In the specific case of the mass density structures of the leptons, there is one temporal parameter for each of the two spatial dimensions. There appears to be no mathematical requirement that these two temporal parameters-dimensions be linked or that one be dependent on the other.

Mathematically there can be a fourth lepton with a positive mass, and thus logically a fourth neutrino and family of quarks. The equations for this fourth lepton were briefly investigated. Although theoretically possible, such a particle appears somewhat improbable. Its mass structure appears inefficient in comparison to the muon and tau. Also its angular geometric appearance indicates that it probably would rapidly self-destruct similar to a badly imbalanced airplane propeller.

One of the factors involved in the scaleup of the mass density equations from math-geometry to the consensus world of physics suggests shielding, binding energy, or the lowering-minimizing of the energy state of a composite structure. That is, in going from the electron to the muon to the tau an effect was found similar to the mass defect found in going from H to He to Li. This implies that the muon and the tau probably have composite internal structures analogous to He and Li.

By rearranging the vector formulation of the structural appearance which gives rise to the leptons' charge, we can back calculate the value of G to three orders of magnitude more accuracy than that to which it can be measured. Although this rearrangement, interchanging of parametric dependencies, runs against the grain of the logic in this work, such a reordering of variables from $e = F_1(G, \mu_0, \epsilon_0)$ to become $G = F_2(e, \mu_0, \epsilon_0)$ is quite simple and legitimate mathematically and serves a beneficial purpose.

VII. ACKNOWLEDGMENTS

We wish to thank Jolanta Pyra for her insightful and incredibly accurate inputs, without which this project would not have been completed. Harold Wright deserves a round of applause for helping turn this work into something understandable for the scientific reader.

VIII. NOMENCLATURE

The following nomenclature is used in this article. It is included here for ease of understanding.

A = Charge correlation constant, C^2 / meter abs
C_g = General mass correlation constant, meter abs
C_p = Individual particle constant, unitless
$C_r k$ = Initial radial constant for shell k
$C_{\theta k}$ = Initial angular constant for shell k
D_p = Mass density of particle, kg/radial meter abs
$D_k(r)$ = Radial mass density of particle shell k
$D_k(\theta)$ = Angular mass density of particle shell k
F_c = Constant factor for particle
F_{dfn} = The Fraunhofer Diffraction Function
F_m = Series member factor for particle
F_s = Shielding or mass defect factor for particle
$I(r)$ = Initial radial condition
$I(\theta)$ = Initial angular condition
$L_n^d(F(r))$ = the dth derivative of the nth Laguerre Orthogonal Polynomial
M_p = mass of particle, kg
r = Radial parameter of mass density expressions
S_k = Shell correlation constant, unitless
t = Generic temporal variable of vector charge expressions
$T_n^\dagger(F(\theta))$ = nth Chebyshev Orthogonal T Polynomial, shifted
t_r = Radial temporal variable of mass density expressions
t_θ = Angular temporal variable of mass density expressions
ρ = Radial parameter of vector charge expressions
θ = Angular parameter of mass density expressions

IX. APPENDIX

X. REFERENCES

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TABLE VIII: **Basic Physical Constants Used in This work [32]**

FUNDAMENTALS, a-priori	UNITS, RELATIVE	NUMERICAL	ERROR
G, gravitational constant	m / kg (m/s)^2	$6.672, 59(10^{-11})$	$8.5(10^{-15})$
μ_0, magnetic constant	(kg m) / C^2	$1.256, 637, 061(10^{-6})$	0
ϵ_0, electrical constant	C^2 / (kg m) (s/m)^2	$8.854, 187, 817(10^{-12})$	0
DERIVABLE, but used as a-priori			
e, electron charge	C	$1.602, 177, 33(10^{-19})$	$4.9(10^{-26})$
α, fine structure constant	unitless	$7.297, 353, 08(10^{-3})$	$3.3(10^{-10})$
DERIVATION OBJECTIVES			
electron mass	kg	$9.109, 389, 7(10^{-31})$	$5.4(10^{-37})$
	MeV/c^2	$0.510, 999, 06$	$1.1(10^{-7})$
muon mass	kg	$1.883, 532, 7(10^{-28})$	$1.1(10^{-34})$
	MeV/c^2	$105.658, 389$	$3.4(10^{-3})$
tau mass	kg	$3.167, 88(10^{-27})$	$5.2(10^{-31})$
	MeV/c^2	$1, 777.05$	$+0.29, -0.26$
CALCULATED for scaling	FORMULAS	NUMERICAL	
G	see paper	$6.672, 590, 32(10^{-11})$	
meter, absolute	$e\mu_0(G\epsilon_0)^{1/2}$	$4.893, 752, 96(10^{-36})$	
MeV / kg	$1/(10^6\mu_0\epsilon_0 e)$	$5.609, 586, 16(10^{29})$	
meter × MeV / kg	$(G/\epsilon_0)^{1/2}/10^6$	$2.745, 192, 89(10^{-06})$	
$1/(2\alpha)$	$1/(2\alpha)$	$6.851, 799, 475(10^1)$	

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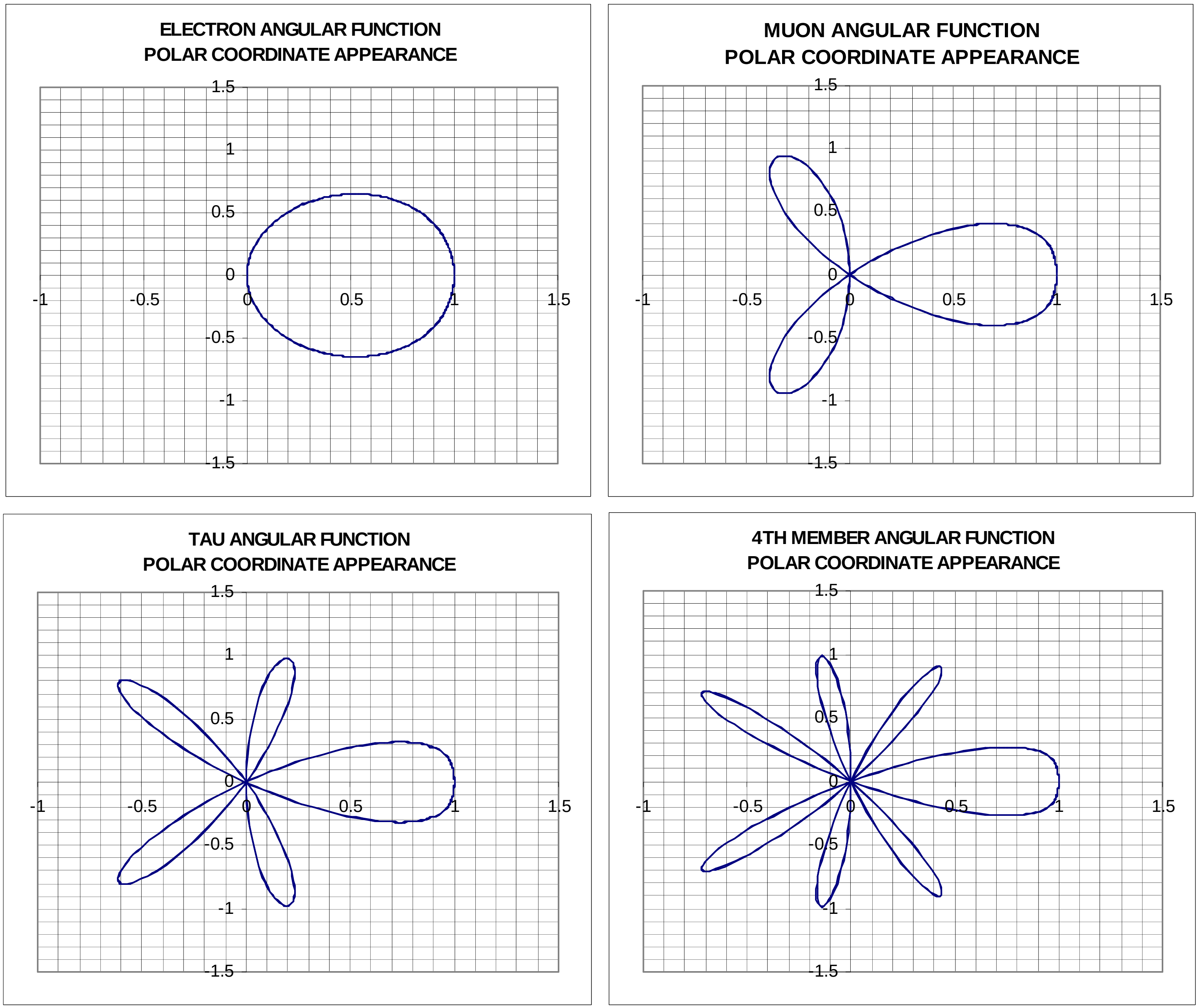
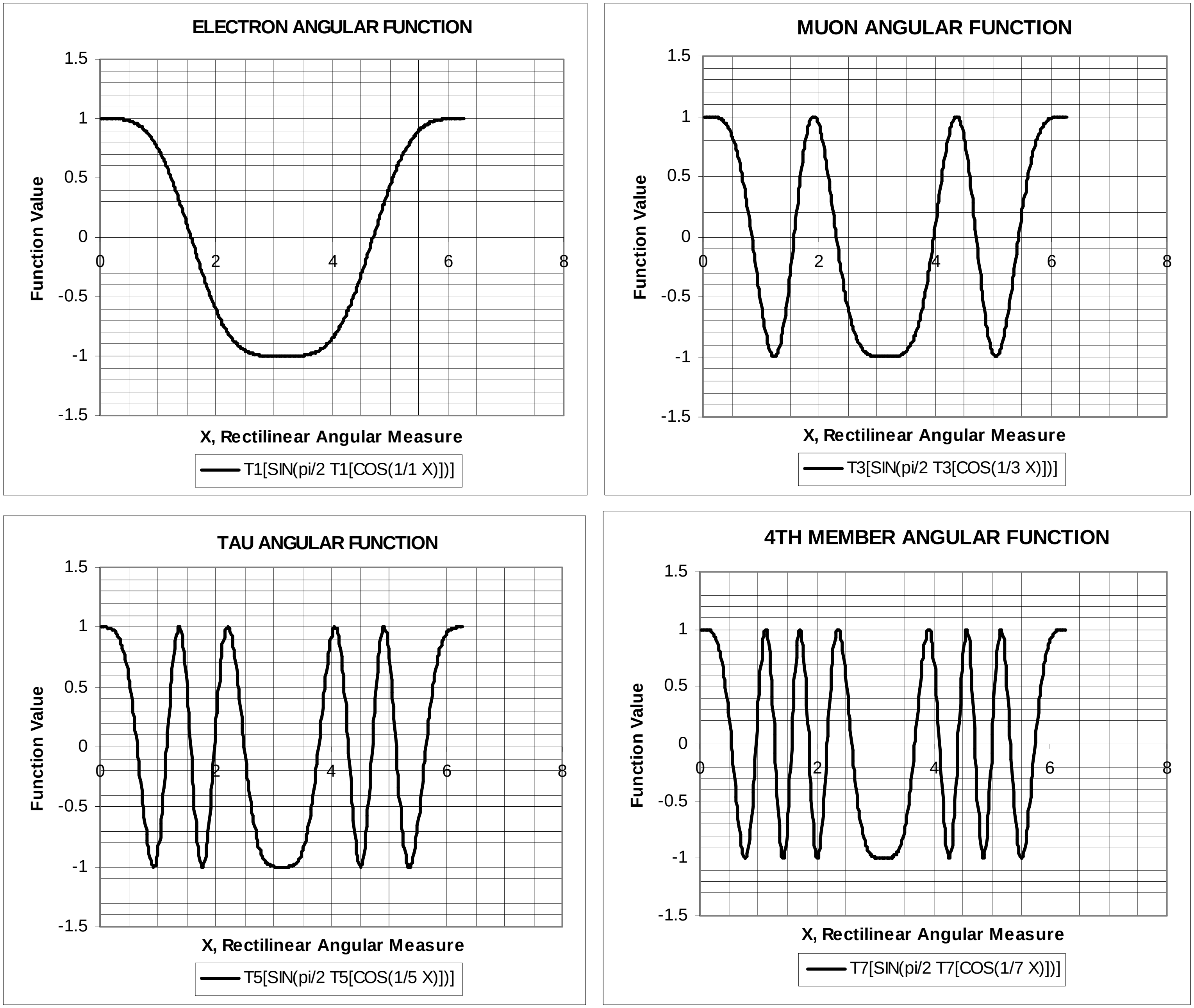
FIGURE 1 a - d



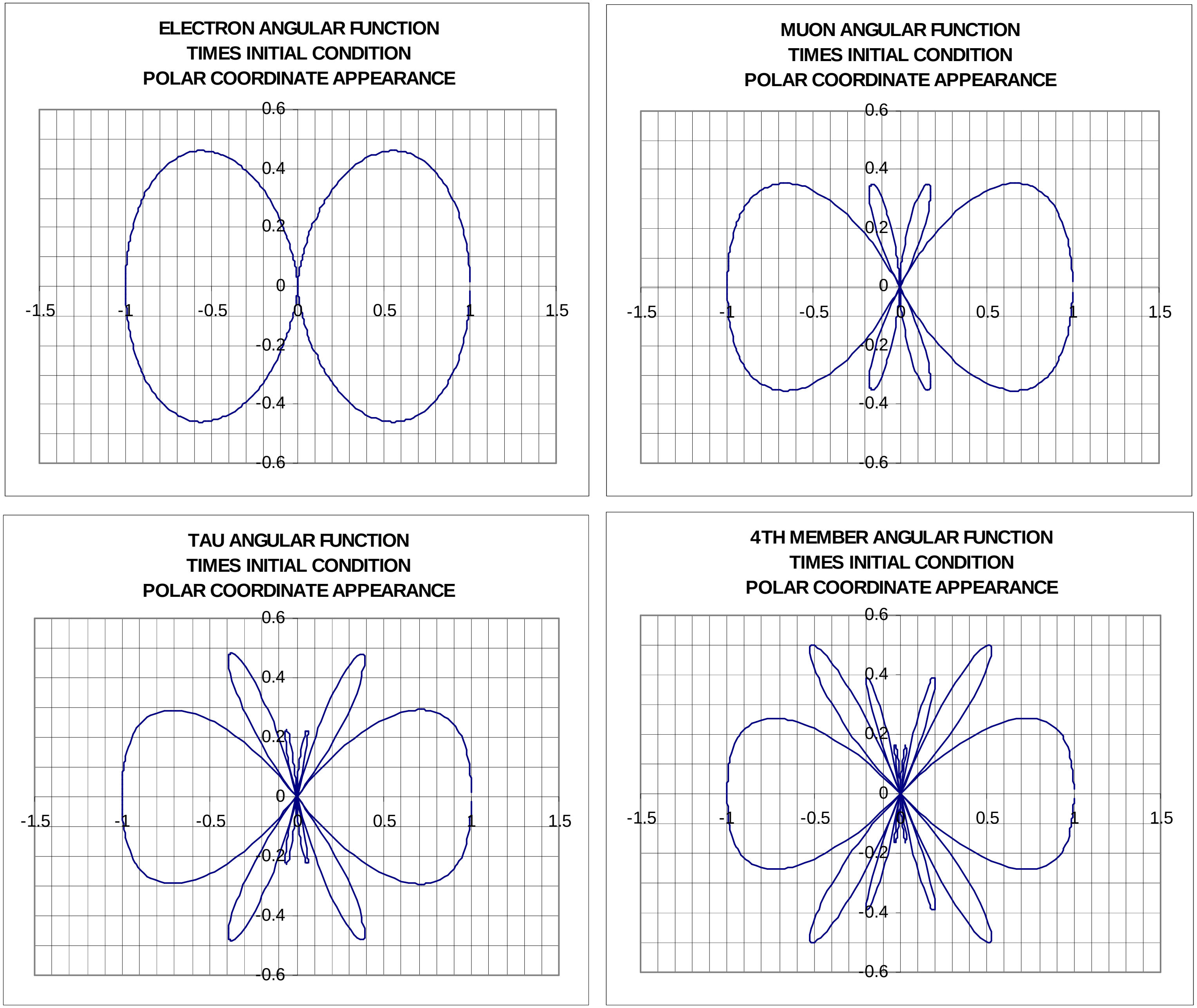
See Sections IV. B.– E., V., and VI. B. in the text for detailed discussions.

FIGURE 2 a - d



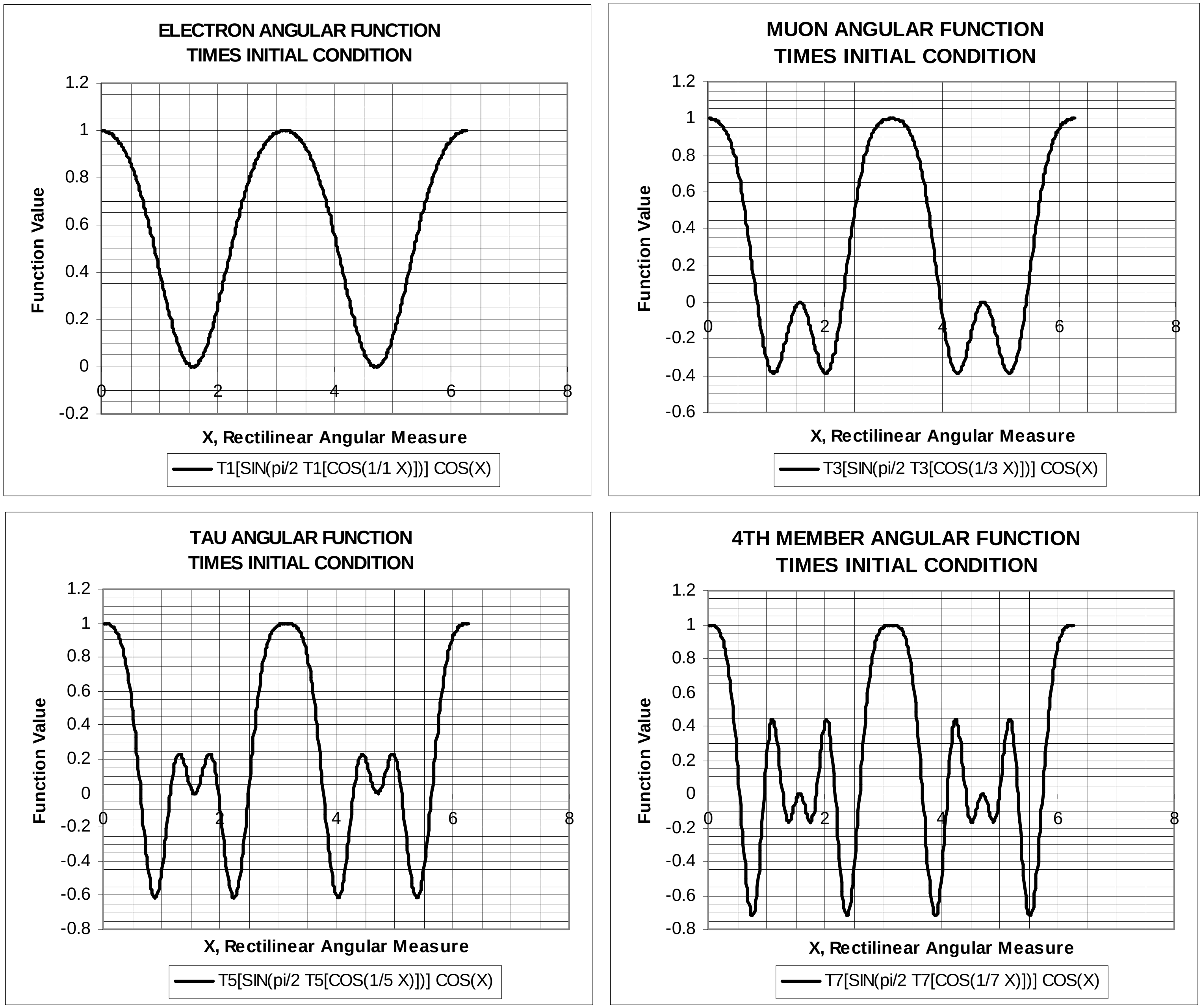
See Sections IV. B.– E., V., and VI. B. in the text for detailed discussions.

FIGURE 3 a – d



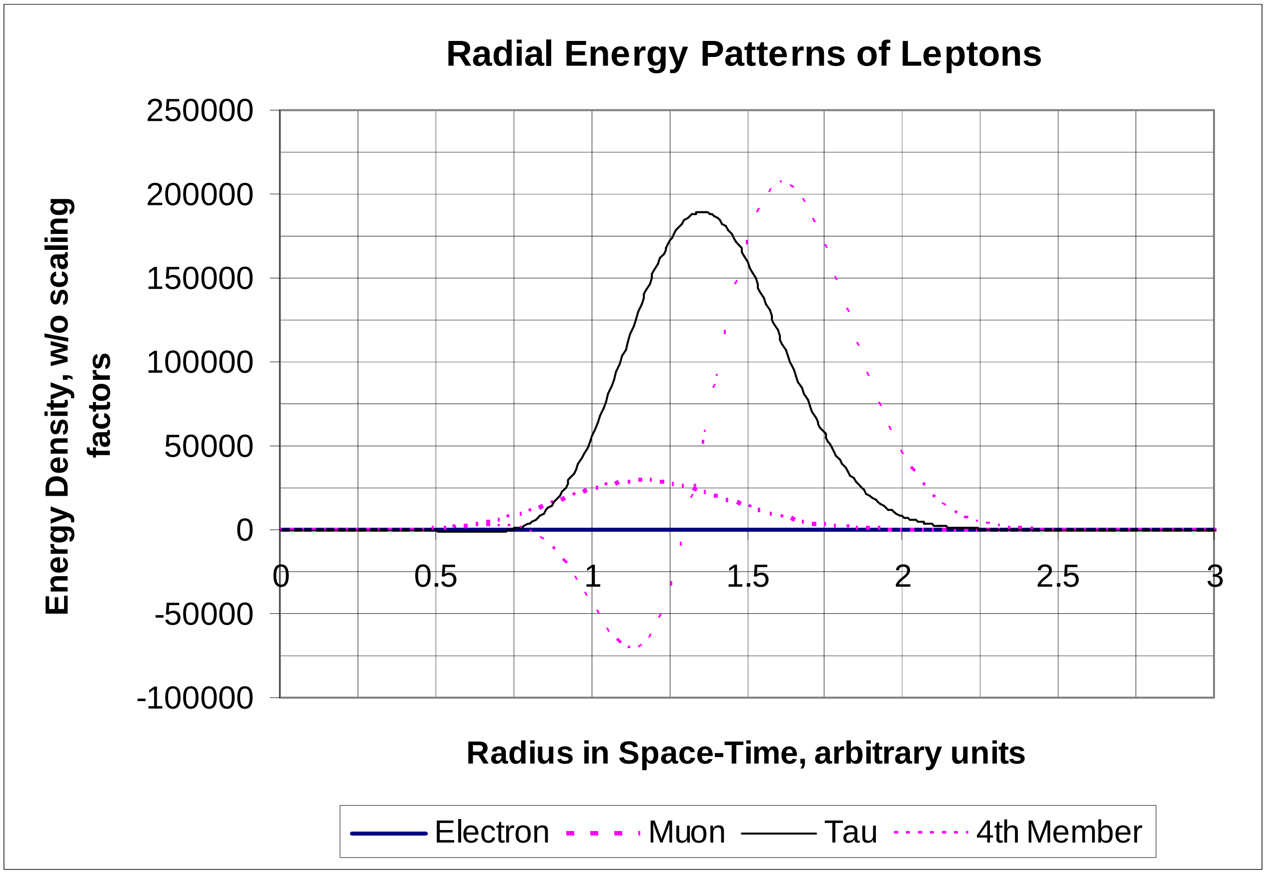
See Sections IV. B.– E., V., and VI. B. in the text for detailed discussions.

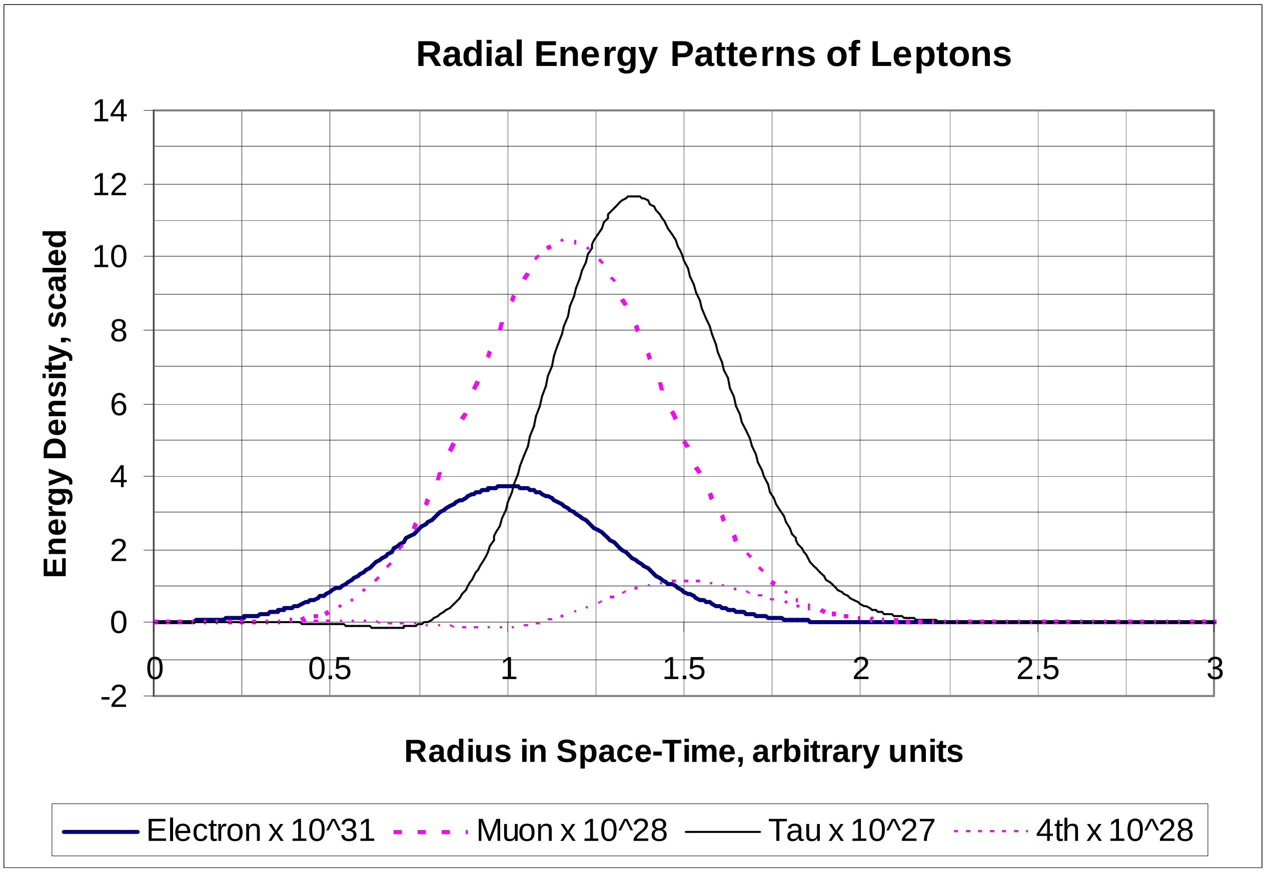
FIGURE 4 a – d



See Sections IV. B.– E., V., and VI. B. in the text for detailed discussions.

FIGURE 5 a & b





See Sections IV. B.– E., V., and VI. C. in the text for detailed discussions.